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Absolute Stability in Delay Equations

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An equilibrium of a delay equation is said to be absolutely stable if it is asymptotically stable for all delays. This is equivalent to a certain characteristic equation's having all its roots in the left half-plane for all delays. We obtain a simple necessary and sufficient condition for absolute stability of a class of delay equations. © 1987 Academic Press, Inc.

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A standard technique for analyzing the stability of an equilibrium of a delay equation or system begins with linearization about the equilibrium. Then a characteristic equation is constructed, expressing the condition that the linearized system have a solution each component of which is a constant multiple of e^{zt}. The condition for stability of the equilibrium is that all roots of the characteristic equation have negative real part. A fixed delay τ is reflected in the characteristic equation by terms $e^{-z\tau}$. Thus the characteristic equation frequently takes the form

$$P(z)+Q(z)\,e^{-z\tau}=0. \tag{1}$$

For a differential-difference equation of retarded type of order n, the characteristic equation is of the form (1) with $P(z)$ and $Q(z)$ polynomials and $P(z)$ having higher degree than $Q(z)$. For a linear system of differential-difference equations

$$u'(t)=Au(t)+Bu(t-\tau), \tag{2}$$

where u is a vector and A and B are constant square matrices, the characteristic equation is

$$\det[A+Be^{-z\tau}-zI]=0,$$

which is of the form (1) if delay enters the system in such a way that expansion of the determinant does not produce products of exponential terms. In

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many applications, the matrix B has few enough nonzero terms that this is the case.

Another example is the delayed nonlinear renewal equation

$$x(t) = \int_0^\infty g\{x(t-r-s)\}\, a(s)\, ds \tag{3}$$

analyzed in [3]. The linearization at an equilibrium x_∞ is

$$u(t) = g'(x_\infty) \int_0^\infty u(t-\tau-s)\, a(s)\, ds,$$

with characteristic equation

$$e^{-\tau z} g'(x_\infty) \int_0^\infty e^{-zs} a(s)\, ds = 1, \tag{4}$$

of the form (1) with

$$P(z) = 1, \qquad Q(z) = -g'(x_\infty) \int_0^\infty e^{-zs} a(s)\, ds.$$

In each of these examples the functions $P(z)$ and $Q(z)$ are analytic in the right half-plane $\mathscr{R}z \geqslant 0$ and $Q(z)/P(z) \to 0$ as $|z| \to \infty$ with $\mathscr{R}z \geqslant 0$.

There are many results giving conditions under which all roots of (1) lie in the left half-plane for particular cases of functions P and Q. A recent result of Cooke and Van den Driessche [7] for a general class of equations (1) considers the location of the roots of (1) as a function of the parameter τ, $0 \leqslant \tau < \infty$, and shows that under the hypotheses (6), (7), (8) below there are the following three possibilities:

(i) The equation (1) has a root in $\mathscr{R}z \geqslant 0$ for all τ, $0 \leqslant \tau < \infty$.

(ii) There is a finite number of values of τ for which roots of (1) may enter or leave $\mathscr{R}z \geqslant 0$, but for all sufficiently large τ the equation (1) has a root in $Rz \geqslant 0$.

(iii) All roots of (1) are in $\mathscr{R}z < 0$ for $0 \leqslant \tau < \infty$.

In the case (iii), the equilibrium whose linearization leads to the characteristic equation (1) is asymptotically stable for all delays τ. Such behavior has been called absolute stability of (1) in [8]. An early general result of Chin [6] is that the equation (2) exhibits absolute stability if and only if the eigenvalues of the matrix $(A + B)$ all have negative real part and

$$\det[A + Be^{iy\tau} - iyI] \neq 0 \tag{5}$$

for $0 \leqslant y < \infty$, $0 \leqslant \tau < \infty$.

Our main result is that under the conditions

$$P(z) \neq 0, \qquad \mathscr{R}z \geqslant 0 \tag{6}$$

$$\overline{P(-iy)} = P(iy), \qquad \overline{Q(-iy)} = Q(iy), \qquad 0 \leqslant y < \infty \tag{7}$$

$$|Q(iy)| < |P(iy)|, \qquad 0 \leqslant y < \infty \tag{8}$$

$$\lim_{|z| \to \infty, \mathscr{R}z \geqslant 0} |Q(z)/P(z)| = 0 \tag{9}$$

the equation (1) is absolutely stable. The proof requires only the result of [7] together with some elementary complex analysis. We shall also give an extension which applies when $P(0) + Q(0) = 0$, a case which arises in many applications but for which (8) is not satisfied.

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We begin this section with our main result.

THEOREM 1. *Let $P(z)$ and $Q(z)$ be analytic in some open set containing $\mathscr{R}z \geqslant 0$ and satisfy the conditions* (6)–(9). *Then the equation* (1) *is absolutely stable.*

Proof. Since $P(z)$ has no zeros in $\mathscr{R}z \geqslant 0$, the function $Q(z)/P(z)$ is analytic in $\mathscr{R}z \geqslant 0$. On a suficiently large semi-circle $|z| = R$ in $\mathscr{R}z \geqslant 0$, $|Q(z)/P(z)| \leqslant \rho < 1$ because of (9), and $|Q(z)/P(z)| \leqslant \rho < 1$ on the line segment $z = iy$, $-R \leqslant y \leqslant R$ because of (8). Now, by the maximum modulus principle, $|Q(z)/P(z)| < 1$ in every large semi-circle in the right half-plane. But $|e^{\tau z}| \geqslant 1$ in the right half-plane, and since (1) is equivalent to

$$-\frac{Q(z)}{P(z)} = e^{\tau z},$$

there can be no root of (1) in the right half-plane for any $\tau \geqslant 0$.

For the characteristic equation $(a - z) + be^{-\tau z} = 0$, where $P(z) = a - z$, $Q(z) = b$, the condition that $P(z)$ have no zeros in $\mathscr{R}z \geqslant 0$ is $a < 0$, and the condition $|Q(iy)| < |P(iy)|$ is $b^2 < a^2 + y^2$ for $0 \leqslant y < \infty$. Thus this equation is absolutely stable if $a < 0$ and $|b| < |a|$. This result is known; see, for example, [8], pp. 130–131]. However, even though this equation has been cited frequently in the literature on population biology, e.g., [10, p. 95] for $a = 0$, [5, 2], this absolute stability result has not become well known. For the delayed nonlinear renewal equation (3) with characteristic equation (4), it follows immediately from Theorem 1 that the condition for absolute stability is $|g'(x_\infty)| \int_0^\infty a(s)\,ds < 1$ [3]. For the characteristic equation of a linear differential-difference equation of order n, the result of Theorem 1 is a theorem of Repin [12].

We shall now consider two examples, formulated in [1, pp. 446–450]. Both will lead to the question of positivity of a quartic polynomial, and we will make use of the observation that the polynomial

$$F(y) = y^4 + \beta y^2 + \gamma$$

is positive for $0 \leqslant y < \infty$ if and only if $\gamma > 0$ and either $\beta^2 - 4\gamma < 0$ or $\beta^2 - 4\gamma \geqslant 0$ and $\beta > 0$.

Our first example concerns the equation

$$(z^2 + az + b) + ce^{-\tau z} = 0, \tag{10}$$

with $P(z) = z^2 + az + b$, $Q(z) = c$. The condition that $P(z) \neq 0$ for $\mathscr{R}z \geqslant 0$ is $a > 0$, $b > 0$. Since $P(iy) = (b - y^2) + iay$, $Q(iy) = c$, the condition $|P(iy)| > |Q(iy)|$ is $(b - y^2)^2 + a^2y^2 > c^2$, which reduces to

$$y^4 + (a^2 - 2b)\,y^2 + (b^2 - c^2) > 0. \tag{11}$$

The condition (11) is satisfied if $b^2 - c^2 > 0$ and either $a^4 + 4c^2 < 4a^2b$ or $a^4 + 4c^2 \geqslant 4a^2b$ and $a^2 > 2b$. Thus the equation (10) is absolutely stable if and only if either $a > 0$, $b > 0$, $b^2 > c^2$, and $a^4 + 4c^2 < 4a^2b$ or $a > 0$, $b > 0$, $b^2 > c^2$, $a^4 + 4c^2 \geqslant 4a^2b$, and $a^2 > 2b$.

Our second example concerns the equation

$$(z^2 + az + b) + cze^{-\tau z} = 0 \tag{12}$$

with $P(z) = z^2 + az + b$, $Q(z) = cz$. The condition that $P(z) \neq 0$ for $\mathscr{R}z \geqslant 0$ is $a > 0$, $b > 0$. Since $P(y) = (b - y^2) + iay$, $Q(iy) = icy$, the condition $|P(iy)| > |Q(iy)|$ is $(b - y^2)^2 + a^2y^2 > c^2y^2$, which reduces to

$$y^4 + (a^2 - c^2 - 2b)y^2 + b^2 > 0.$$

This condition is satisfied if either

$$(a^2 - c^2 - 2b)^2 - 4b^2 = (a^2 - c^2)(a^2 - c^2 - 4b) < 0 \tag{13}$$

or

$$(a^2 - c^2)(a^2 - c^2 - 4b) > 0 \qquad \text{and} \qquad a^2 - c^2 - 2b > 0. \tag{14}$$

If $b > 0$ then $a^2 - c^2 > a^2 - c^2 - 4b$, and (13) can be satisfied only if $a^2 - c^2 > 0$, $a^2 - c^2 - 4b < 0$. The condition (14) is satisfied if and only if $a^2 - c^2 - 4b > 0$. Thus the equation (12) is absolutely stable if and only if

$a>0$, $b>0$ and either $0<a^2-c^2<4b$ or $a^2-c^2>4b$, that is if and only if $a>0$, $b>0$, $a^2>c^2$.

The conditions (6) and (8) are necessary for absolute stability in the sense that if (8) is violated it has been shown in [7] that there are values of τ for which (1) has roots in $\mathscr{R}z\geqslant 0$ and if (8) and (9) hold but (6) is violated, then in any large semi-circle in the right half-plane,

$$|Q(z)\,e^{-\tau z}|\leqslant|Q(z)|<|P(z)|,$$

and Rouché's theorem shows that $P(z)$ and $P(z)+Q(z)\,e^{-\tau z}$ have the same number of zeros in this semi-circle, so that (1) can not be absolutely stable. On the other hand, the condition (9) is not necessary for absolute stability, as may be seen immediately by taking $P(z)=cQ(z)$ with $c>1$.

The argument used in Theorem 1 can be extended immediately to the characteristic equation

$$P(z)+\sum_{j=1}^{n} Q_j(z)\,e^{-\tau_j z}=0,$$

where $0<\tau_1<\tau_2<\cdots<\tau_n$ under the conditions (6),

$$\sum_{j=1}^{n}|Q_j(iy)|<|P(iy)|,\qquad 0\leqslant y<\infty \tag{15}$$

$$\lim_{|z|\to\infty,\,\mathscr{R}z\geqslant 0}\ \sum_{j=1}^{n}|Q_j(z)/P(z)|=0. \tag{16}$$

This gives an absolute stability condition for first order differential-difference equations with several delays.

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The condition $P(0)+Q(0)\neq 0$ in Theorem 1 implies that $z=0$ is not a root of (1) for any $\tau\geqslant 0$; if $z=0$ is a root of (1) then the equilibrium of the delay equation which leads to the characteristic equation (1) is not asymptotically stable. However, in some cases a characteristic equation may contain a factor z because it has been derived from an insufficiently specified problem. An integration plus an axiliary condition, which would have the effect of removing this factor of z, is needed to describe the problem completely. An example is the characteristic equation

$$z^2+cz+(1-e^{-z\tau})(az+b)=0 \tag{17}$$

derived in [4] to describe the variable maturation time stage-structure population model of Nisbet and Gurney [11]; see also [9]. Here

$$P(z) = z^2 + (a+c)z + b, \qquad Q(z) = -az - b,$$

and $P(0) + Q(0) = 0$. However, the proper form of the characteristic equation (17) is

$$z + c + \left(\frac{1 - e^{-z\tau}}{z}\right)(az + b) = 0; \tag{18}$$

observe that $(1 - e^{-z\tau})/z$ is analytic for $\mathscr{R}z \geqslant 0$, but (18) is not of the form (1). In order to analyze the equation (17), and other equations which have been put in the form (1) by multiplication by a spurious factor z, it would be useful to have an absolute stability result for (1) with $P(0) + Q(0) = 0$.

THEOREM 2. *Suppose that $P(z)$ and $Q(z)$ are analytic in some open set containing $z \geqslant 0$, and satisfy the following conditions:*

(i) $P(z) \neq 0$, $\mathscr{R}z \geqslant 0$.
(ii) $\overline{P(-iy)} = P(iy)$, $\overline{Q(-iy)} = Q(iy)$, $0 \leqslant y < \infty$.
(iii) $P(0) + Q(0) = 0$.
(iv) $|Q(iy)| < |P(iy)|$ *for* $0 < y < \infty$.
(v) $\lim_{|z| \to \infty, \mathscr{R}z \geqslant 0} |Q(z)/P(z)| = 0$.

Then except for the root $z = 0$, all roots of (1) are in $\mathscr{R}z < 0$ for $0 \leqslant \tau < \infty$.

Proof. We apply Theorem 1 to the equation

$$P(z) + \varepsilon P(z) + Q(z)\, e^{-\tau z} = 0 \tag{19}$$

for sufficiently small $\varepsilon > 0$. Since

$$P(0) + \varepsilon P(0) + Q(0) = \varepsilon P(0) \neq 0$$

and

$$|Q(0)| = |P(0)| < (1 + \varepsilon)\,|P(0)|,$$

the hypotheses of Theorem 1 are satisfied for all small $\varepsilon > 0$. Thus all roots of (19) lie in $\mathscr{R}z < 0$ for sufficiently small $\varepsilon > 0$. Because the roots of (19) depend continuously on ε, the roots of (1) must lie in $\mathscr{R}z \leqslant 0$. However, the condition (iv) guarantees that there are no roots of (1) of the form $z = iy$ with $y > 0$. Thus except for the root $z = 0$, all roots of (1) are in $\mathscr{R}z < 0$.

For the equation (17), the condition that $P(z) \neq 0$ for $\mathscr{R}z \geqslant 0$ is $a + c > 0$, $b > 0$. Since $|P(iy)|^2 = (-y^2 + b)^2 + (a+c)^2 b^2$ and $|Q(iy)|^2 = b^2 + a^2 y^2$, the condition $|Q(iy)| < |P(iy)|$ for $y>0$ is $y^4 + y^2(2ac + c^2 - 2b) > 0$ for $y>0$, or $b < ac + c^2/2$. Thus, except for the (simple) root $z=0$, all roots of (17) are in $\mathscr{R}z < 0$ for $0 \leqslant \tau < \infty$ provided $a+c>0$, $0<b<ac+c^2/2$. This gives an absolute stability condition for the equation (17).

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